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TECHNICAL NOTES

A note on two-dimensional linearized perturbations of the Neumann problem

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NOMENCLATURE

c	specific heat
C, $\bar{C}$	interface functions
L	latent heat of fusion
t	time
T	temperature
T_0	boundary temperature
T_F	fusion temperature
x, y	Cartesian coordinates.

Greek symbols

β	Stefan number
θ	dimensionless temperature
θ_1	dimensionless temperature perturbation
κ	thermal diffusivity
λ	Neumann constant
ξ, η	dimensionless similarity variables
ϕ_1	interface perturbation function.

1. INTRODUCTION

ANALYTICAL solutions to the Stefan problem involving more than one space variable are few and far between. However, there is a class of such problems (usually where the geometry and the boundary conditions are such that no natural independent length scale is available) which approach the one-dimensional (1-D) Neumann solution in some asymptotic limit. An example is the problem of solidification in a right-angled corner (the quarter-infinite region) where, far away from the corner itself, the Neumann solution would eventually evolve (see Fig. 1). In fact the same situation would occur in the case of an infinite corner region of arbitrary angle. However, for definiteness, consider for the moment the Stefan problem of solidification of a quarter-infinite region of liquid ($x > 0$, $y > 0$) initially at its fusion temperature T_F, when the bounding surfaces ($x = 0$, $y = 0$) are maintained at a constant temperature $T_0 < T_F$. Let L be the latent heat of fusion of the solidifying substance, and c be its specific heat. Let κ be the thermal diffusivity in the solid phase. Let $\bar{C}(x, y, t) = 0$ be a function which represents the boundary between the liquid and the solid at time t. Then, for example, for large y, we expect the solution to approach the classical 1-D Neumann solution for the half-space (with a corresponding situation for large x). In Fig. 1, the dotted lines represent the Neumann asymptotes to the solid–liquid interface. In this note, we aim to find the leading term in the asymptotic approach to the Neumann form, by consideration of linearized perturbations about the latter. (Of course this particular example is clearly only one of a large class of such possible problems.)

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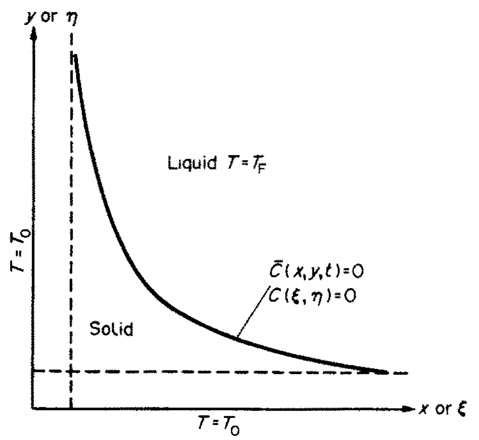

FIG. 1. Representative sketch of typical problem.

2. FUNDAMENTAL EQUATIONS

For the type of problem described above, there is no natural independent length scale, and it is easily shown therefore, on dimensional grounds, that x, y and t occur only in the combination

$$\xi = x/2\surd(\kappa t), \quad \eta = y/2\surd(\kappa t). \tag{1}$$

Let the temperature in the solid region be T, and define

$$\theta = (T - T_0)/(T_F - T_0) \tag{2}$$

$$\beta = L/c(T_F - T_0) \tag{3}$$

in the usual way. Let the equation of the solid–liquid interface in the (ξ, η) plane be represented by $C(\xi, \eta) = 0$. Then the governing equations in these variables are

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + 2\xi\frac{\partial\theta}{\partial\xi} + 2\eta\frac{\partial\theta}{\partial\eta} = 0 \tag{4}$$

$$\theta = 0 \quad \text{on the external boundaries} \tag{5}$$

$$\theta = 1 \quad \text{on} \quad C(\xi, \eta) = 0 \tag{6}$$

and

$$\frac{\partial\theta}{\partial\xi}\frac{\partial C}{\partial\xi} + \frac{\partial\theta}{\partial\eta}\frac{\partial C}{\partial\eta} - 2\beta\left(\xi\frac{\partial C}{\partial\xi} + \eta\frac{\partial C}{\partial\eta}\right) = 0 \ \text{on} \ C(\xi, \eta) = 0. \tag{7}$$

Equation (4) is simply the heat conduction equation, and equation (7) is the usual 'latent heat' condition at the solid–liquid boundary, expressed in the appropriate variables [1].

Suppose now that the coordinates are chosen so that $\eta = 0$ is a plane external boundary, and that the problem is such that the Neumann solution is approached as $\eta \to \infty$. The

Neumann solution is simply

$$\theta = \text{erf}\ \xi/\text{erf}\ \lambda, \quad C = \xi - \lambda \tag{8}$$

where λ is the unique real root of the transcendental equation

$$\lambda\ e^{\lambda^2}\ \text{erf}\ \lambda = 1/(\beta\sqrt{\pi}). \tag{9}$$

The approach of this note is to write

$$\begin{aligned}\theta &= \text{erf}\ \xi/\text{erf}\ \lambda + \theta_1(\xi, \eta)\\ C &= (\xi-\lambda) - \phi_1(\eta)\end{aligned} \tag{10}$$

where θ_1 and ϕ_1 tend to zero as $\eta \to \infty$. However, it is convenient first to obtain some solutions of equation (4) (which is of course satisfied by θ_1). Seeking a solution $\theta_1 = f(\xi)g(\eta)$ by separation of variables, one obtains

$$\begin{aligned}f'' + 2\xi f' - 2\mu f &= 0\\ g'' + 2\eta g' + 2\mu g &= 0.\end{aligned} \tag{11}$$

If $\mu = 0$, the solutions for f and g are just error functions. For $\mu = n+1$, $n = 0, 1, 2, \ldots$, one obtains that

$$f_n = A_n\ i^{n+1}\ \text{erfc}\ \xi + C_n\ i^{n+1}\ \text{erfc}\ (-\xi) \tag{12}$$

and

$$g_n = e^{-\eta^2} H_n(\eta) \quad \text{(going to zero as } \eta \to \infty). \tag{13}$$

Here A_n and C_n are constants, and H_n is the Hermite polynomial of degree n. Clearly for negative values of μ, the forms of the solutions for f and g will be interchanged. However, it will become clear later that a complete set of orthogonal eigenfunctions in η is needed, so that positive integral values of μ (together with $\mu = 0$) can be used. Thus a solution satisfying $\theta_1 = 0$ on $\xi = 0$, and $\theta_1 \to 0$ as $\eta \to \infty$ can be obtained in the form

$$\theta_1 = \alpha\ \text{erf}\ \xi\ \text{erfc}\ \eta + \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta) \times [i^{n+1}\ \text{erfc}\ \xi - i^{n+1}\ \text{erfc}\ (-\xi)] \tag{14}$$

where α and A_n are constants.

One needs to linearize the position at which the remaining boundary conditions are applied, from $\xi = \lambda + \phi_1(\eta)$ to $\xi = \lambda$, in the usual way using Taylor's theorem, to obtain

$$\left.\left(\theta + \phi_1 \frac{\partial \theta}{\partial \xi}\right)\right|_{\xi=\lambda} = 1 \tag{15}$$

$$\left.\left(\frac{\partial \theta}{\partial \xi} + \phi_1 \frac{\partial^2 \theta}{\partial \xi^2} - \phi_1' \frac{\partial \theta}{\partial \eta} - \phi_1 \phi_1' \frac{\partial^2 \theta}{\partial \xi\, \partial \eta} - 2\beta[\lambda + \phi_1 - \eta\phi_1']\right)\right|_{\xi=\lambda} = 0 \tag{16}$$

where θ will be given by equation (10).

One can note the following

$$\theta|_{\xi=\lambda} = 1 + \alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta + \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta)[i^{n+1}\ \text{erfc}\ \lambda - i^{n+1}\ \text{erfc}\ (-\lambda)] \tag{17}$$

$$\left.\frac{\partial \theta}{\partial \xi}\right|_{\xi=\lambda} = 2\beta\lambda(1 + \alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta) - \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta)[i^n\ \text{erfc}\ \lambda + i^n\ \text{erfc}\ (-\lambda)] \tag{18}$$

$$\left.\frac{\partial^2 \theta}{\partial \xi^2}\right|_{\xi=\lambda} = -4\beta\lambda^2(1 + \alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta) + \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta)[i^{n-1}\ \text{erfc}\ \lambda - i^{n-1}\ \text{erfc}\ (-\lambda)]. \tag{19}$$

(Here i^{-1} erfc X is interpreted as $-(2/\sqrt{\pi})\ e^{-X^2}$, and equation (9) is used.) Note also that

$$\left.\frac{\partial \theta}{\partial \eta}\right|_{\xi=\lambda} \text{ and } \left.\frac{\partial^2 \theta}{\partial \xi\, \partial \eta}\right|_{\xi=\lambda} \text{ are } O(e^{-\eta^2})^* \text{ as } \eta \to \infty. \tag{20}$$

Hence equation (15) becomes

$$\begin{aligned}&1 + \alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta + \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta) \times [i^{n+1}\ \text{erfc}\ \lambda - i^{n+1}\ \text{erfc}\ (-\lambda)]\\ &+ \phi_1\left\{2\beta\lambda + 2\beta\lambda\ \text{erf}\ \lambda\ \text{erfc}\ \eta - \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta)[i^n\ \text{erfc}\ \lambda + i^n\ \text{erfc}\ (-\lambda)]\right\} = 1.\end{aligned} \tag{21}$$

One can see that, as expected, $\phi_1 = O(e^{-\eta^2})$ as $\eta \to \infty$, so that, on a theory linear in $\varepsilon = e^{-\eta^2}$, all terms in the brace bracket except the first may be neglected, since they also are of $O(\varepsilon)$ as $\eta \to \infty$. Thus, on a linearized theory

$$\phi_1 = -\frac{1}{2\beta\lambda}\left[\alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta + \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta) \times [i^{n+1}\ \text{erfc}\ \lambda - i^{n+1}\ \text{erfc}\ (-\lambda)]\right]. \tag{22}$$

3. DETERMINATION OF THE A_n

The remaining condition to be satisfied is equation (16). Clearly one can neglect the terms involving η-derivatives. If the expressions for θ and ϕ_1 are substituted into equation (16), and the standard properties of the Hermite polynomials, namely

$$\frac{d}{d\eta}[e^{-\eta^2} H_n(\eta)] = -e^{-\eta^2} H_{n+1}(\eta) \tag{23}$$

and

$$2\eta H_{n+1}(\eta) = H_{n+2}(\eta) + 2(n+1)H_n(\eta) \tag{24}$$

are used, then, after some manipulation, one obtains

$$\begin{aligned}&\left(2\beta\lambda + 2\lambda + \frac{1}{\lambda}\right)\alpha\ \text{erf}\ \lambda\ \text{erfc}\ \eta + 2\alpha\ \text{erf}\ \lambda\eta\ e^{-\eta^2}/(\lambda\sqrt{\pi})\\ &+ \sum_{n=0}^{\infty}\left(2\lambda + \frac{1}{\lambda}\right) A_n\ e^{-\eta^2} H_n(\eta) \times [i^{n+1}\ \text{erfc}\ \lambda - i^{n+1}\ \text{erfc}\ (-\lambda)]\\ &- \sum_{n=0}^{\infty} A_n\ e^{-\eta^2} H_n(\eta)[i^n\ \text{erfc}\ \lambda + i^n\ \text{erfc}\ (-\lambda)]\\ &+ \frac{1}{2\lambda}\sum_{n=0}^{\infty} A_n\ e^{-\eta^2}\{H_{n+2}(\eta) + 2(n+1)H_n(\eta)\} \times [i^{n+1}\ \text{erfc}\ \lambda - i^{n+1}\ \text{erfc}\ (-\lambda)] = 0.\end{aligned} \tag{25}$$

It remains now to satisfy equation (25) by a suitable choice of A_n. Essentially what is being done here is to choose the A_n by equating coefficients in an asymptotic expansion for large η, but in practice it is much easier to proceed as follows. One attempts to satisfy equation (25) for *all* η, by using the orthogonality relations for the Hermite polynomials. However, there is a slight complication in this procedure, which in fact can be easily circumvented precisely because the results are valid only as $\eta \to \infty$. The orthogonality relation is

$$\int_{-\infty}^{\infty} e^{-\eta^2} H_n(\eta) H_m(\eta)\ d\eta = 2^n n! \sqrt{\pi}\delta_{m,n} \tag{26}$$

and simply multiplying equation (25) by $H_m(\eta)$ and integrating would make the first term (involving erfc η) divergent at minus

* More strictly, $e^{-\eta^2}$ times an algebraic function of η, but this is of little consequence.

infinity. The remedy is to replace

$$\text{erfc } \eta = 1-\text{erf } \eta$$

by

$$(1-\text{erf } \eta)\cdot\tfrac{1}{2}(1+\text{erf } \eta) = \tfrac{1}{2}[1-(\text{erf } \eta)^2] \tag{27}$$

and satisfy this *modified* form of equation (25) for *all* η. Note that this procedure is perfectly legitimate in an approximation *linear* in small perturbations as $\eta \to \infty$, since the difference between the original and modified forms of equation (25) is *quadratically* small in $e^{-\eta^2}$ as $\eta \to \infty$. But now the trouble as $\eta \to -\infty$ is avoided.

Rather than calculate A_n explicitly, it is much easier simply to integrate equation (25) after multiplication by $H_m(\eta)$, using the orthogonality relations. This then leads to a recurrence relation between A_n and A_{n-2}, so that A_n are known recursively, once A_0 and A_1 are calculated explicitly. To do this certain definite integrals are needed, whose evaluation is essentially elementary. Their values are set out in the appendix. One multiplies equation (25), modified by equation (27), respectively by H_0, H_1, and H_2, and use the orthgonality relations together with the integrals in the appendix, to obtain

$$\left\{[\text{erfc } \lambda+\text{erfc } (-\lambda)]-\left(2\lambda+\frac{3}{\lambda}\right)[\text{i erfc } \lambda-\text{i erfc } (-\lambda)]\right\}A_0 = 2^{3/2}\left(\beta\lambda+\lambda+\frac{1}{2\lambda}\right)\alpha \text{ erf } \lambda/\pi \tag{28}$$

$$\left\{[\text{i erfc } \lambda+\text{i erfc } (-\lambda)]-\left(2\lambda+\frac{5}{\lambda}\right)[\text{i}^2 \text{ erfc } \lambda-\text{i}^2 \text{ erfc } (-\lambda)]\right\} \times A_1 = \alpha \text{ erf } \lambda/(\lambda\sqrt{\pi}) \tag{29}$$

and

$$2^n n!\sqrt{\pi}A_n\left\{[\text{i}^n \text{ erfc } \lambda+\text{i}^n \text{ erfc } (-\lambda)] -\left(2\lambda+\frac{(2n+3)}{\lambda}\right)[\text{i}^{n+1} \text{ erfc } \lambda-\text{i}^{n+1} \text{ erfc } (-\lambda)]\right\} -\frac{1}{2\lambda}\cdot 2^{n-2}(n-2)!\sqrt{\pi}A_{n-2}[\text{i}^{n-1} \text{ erfc } \lambda-\text{i}^{n-1} \text{ erfc } (-\lambda)]$$
$$= \begin{cases} \dfrac{(-1)^{n/2}n!(\beta\lambda+\lambda+1/2\lambda)\alpha \text{ erf } \lambda}{(n+1)(n/2)!2^{(n-3)/2}\sqrt{\pi}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{30}$$

Finally, let $A_n = \alpha B_n$. Then the B_n are defined by the recurrence relations, equations (28)–(30) above, but with α formally replaced by unity, and then

$$\theta = \text{erf } \xi/\text{erf } \lambda+\alpha\left\{\text{erf } \xi \text{ erfc } \eta + \sum_{n=0}^{\infty} B_n \, e^{-\eta^2} H_n(\eta)[\text{i}^{n+1} \text{ erfc } \xi-\text{i}^{n+1} \text{ erfc } (-\xi)]\right\} \tag{31}$$

and

$$\phi_1(\eta) = \frac{-\alpha}{2\beta\lambda}\left\{\text{erf } \lambda \text{ erfc } \eta + \sum_{n=0}^{\infty} B_n \, e^{-\eta^2} H_n(\eta)[\text{i}^{n+1} \text{ erfc } \lambda-\text{i}^{n+1} \text{ erfc } (-\lambda)]\right\} \tag{32}$$

where α is arbitrary. It should be stressed that, although the generalized Fourier coefficients B_n satisfy equation (25) modified by equation (27) exactly, the linearization, together with equation (27), means that one has obtained only the leading term in the asymptotic form of the solution as $\eta \to \infty$.

Thus, some arbitrariness is to be expected, since one has only obtained an asymptotic representation for large η, with no 'initial' conditions imposed. The constant α would have to be determined in any particular problem from comparison with a full numerical integration, at suitably large η. In principle one could continue with

$$\begin{aligned} \theta &= \text{erf } \xi/\text{erf } \lambda+\theta_1(\xi,\eta)+\theta_2(\xi,\eta)+\cdots \\ \xi &= \lambda+\phi_1(\eta)+\phi_2(\eta)+\cdots \end{aligned} \tag{33}$$

and more unknown constants would arise. However, the algebra involved would appear to be prohibitive.

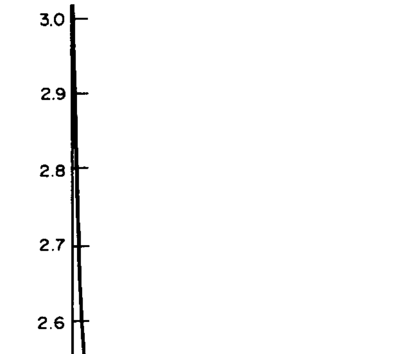

FIG. 2. Comparison between present asymptotic solution and Rathjen and Jiji's numerical solution.

4. COMPARISON WITH NUMERICAL SOLUTION: RIGHT-ANGLED CORNER

Rathjen and Jiji [2] have studied the case of a right-angled corner. They obtained a finite-difference solution, and an approximate analytical solution, based on Lightfoot's [3] method. An integro-differential equation resulted, and in order to make progress with this, they approximated the solution by assuming that the form of the solid–liquid boundary was a 'super-hyperbola' of the form (in the present notation)

$$(\eta^m-\lambda^m)(\xi^m-\lambda^m) = C \tag{34}$$

and obtained good agreement with their finite-difference solution. Some fairly high values of m emerged, which is hardly surprising, since this note shows that the approach to the Neumann solution is exponential, rather than algebraic. Although their results are largely presented graphically, sufficient resolution is available to compare with the present solution.

Since the constant α is just a multiplicative scaling factor, one expects that, for various values of η, a linear correlation be found between $(\xi-\lambda)$ from Rathjen and Jiji and the present $\phi_1(\eta)$ with $\alpha = 1$. The constant of proportionality effectively determines the actual value of α. Now although equations (28)–(32) appear somewhat complex, the calculation is in fact very easy. The integrals of the complementary error function, the Hermite polynomials, and the B_n themselves, can all be calculated iteratively in n, and the series evaluated term by term. For $\beta = 0.1$, it was found that the difference between taking 10 and 20 terms was in the fourth significant figure, so 20 terms were considered sufficient for a rough comparison. For η in the range 2.25(0.05)2.8, a correlation coefficient of 0.999 was found, and the regression gave $\alpha = 0.18(5)$, with a very small intercept at about 2×10^{-5}, which should in theory be zero. Note that taking too large a value of η for comparison gives insufficient difference from the Neumann solution to rely on with any accuracy, and too small a value will be affected by higher order terms in the asymptotic expansion. Given this, the agreement is clearly very satisfactory, and bears out the form of the solution. The results are presented graphically in Fig. 2.

5. CONCLUSIONS

The leading term in the asymptotic approach of a two-dimensional Stefan problem to the Neumann solution has been found, in a form amenable to calculation. Agreement with available numerical results for the case of a right-angled corner is good.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd edn). Oxford University Press, Oxford (1959).
2. K. A. Rathjen and L. M. Jiji, *Trans. Am. Soc. Mech. Engrs, Series C, J. Heat Transfer* 101–109 (1971).
3. N. M. H. Lightfoot, *Proc. London Math. Soc.* **31**, 97–116 (1929).

APPENDIX

The following results are needed in the foregoing analysis. The proofs are straightforward, and so are omitted. If

$$J_n = \int_{-\infty}^{\infty} \eta\, e^{-\eta^2} H_n(\eta)\, d\eta \qquad (A1)$$

then

$$J_1 = \sqrt{\pi}, \quad \text{and} \quad J_n = 0, \quad \text{for } n = 0,2,3,4,\ldots \qquad (A2)$$

If

$$I_n = \int_{-\infty}^{\infty} [1-(\mathrm{erf}\,\eta)^2] H_n(\eta)\, d\eta \qquad (A3)$$

then

$$I_n = 0 \quad \text{if } n \text{ is odd}$$

and

$$I_n = \frac{(-1)^{n/2} n!}{(n+1)(n/2)!\,2^{(n-3)/2}\sqrt{\pi}} \quad \text{if } n \text{ is even.} \qquad (A4)$$

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Heat transfer in the flow of a second-order fluid between two enclosed rotating discs

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NOMENCLATURE

$a_{i,j}$	covariant derivative of the covariant acceleration vector a_i
c_v	specific heat
d_{ij}	strain rate tensor
d_i^m	mixed strain rate tensor
E	Eckert number, $\Omega z_0^2/c_v(T_b - T_a)$
G	dimensionless function of ζ
H	dimensionless function of ζ
k	thermal conductivity
L	dimensionless function of ζ
M	dimensionless function of ζ
Nu_a	average Nusselt number on the lower disc
Nu_b	average Nusselt number on the upper disc
Pr	Prandtl number, $\mu_1 c_v/k$
q_a	heat flux from the lower disc
q_b	heat flux from the upper disc
Q_a	amount of heat transfer from the lower disc
Q_b	amount of heat transfer from the upper disc
r	radial coordinate
Re_l	Reynolds number based on circulatory flow, $l/2\pi\rho z_0 v_1$
Re_m	Reynolds number based on net radial outflow, $m/2\pi\rho z_0 v_1$
Re_z	Reynolds number based on the gap length z_0, $\Omega z_0^2/v_1$
T	temperature
T^*	dimensionless temperature, $Tc_v/v_1\Omega$
T_a	temperature at the lower disc
T_a^*	dimensionless temperature at the lower disc
T_b	temperature at the upper disc
T_b^*	dimensionless temperature at the upper disc
u	radial velocity
U	dimensionless radial velocity, $u/\Omega z_0$
v	azimuthal velocity
$v_{i,j}$	covariant derivative of the covariant velocity vector v_i
v_i^m	covariant derivative of the contravariant velocity vector v^m
V	dimensionless azimuthal velocity, $v/\Omega z_0$
w	axial velocity
W	dimensionless axial velocity, $w/\Omega z_0$
z	axial coordinate
z_0	gap length between the lower and upper discs.